05/03/14

$\mathbb{R}^n$ come spazio metrico $\Rightarrow \exists$ delle distanze:

$$d_p(x,y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p} \qquad 1 \le p < +\infty$$

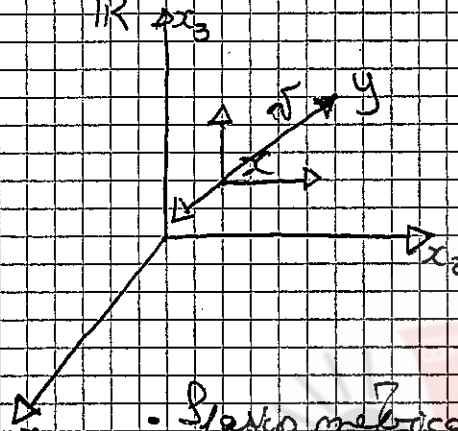
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Se $p=1$ d_1; $p=2$ d_2; $p=\infty$ d_∞. $\forall x,y \in \mathbb{R}^n$

$\exists a,b,c,d$ t.c. $\frac{1}{a} d_\infty(x,y) \le \frac{1}{b} d_1(x,y) \le d_2(x,y) \le c\, d_1(x,y) \le K\, d_\infty(x,y)$

Non sempre l'esistenza di una distanza mi comporta le disuguaglianze precedenti.

Es. $d(x,y) = \#\{i = 1,\dots,n \mid x_i \neq y_i\}$ d non è equivalente a d_p.



$y = x + (y - x)$

v → vettore centrato in x, punta in y.

Per sapere quanto sia lungo v, usiamo le norme.

- Spazio metrico

Def: Una funzione $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ t.c. si dice norma se:

1) $\|v\| \ge 0 \ \forall v \in \mathbb{R}^n$ e $\|v\| = 0 \Leftrightarrow v = 0$
2) $\|v + w\| \le \|v\| + \|w\| \ \forall v, w \in \mathbb{R}^n$
3) $\|\lambda v\| = |\lambda| \cdot \|v\| \ \forall v \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$

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Esempi

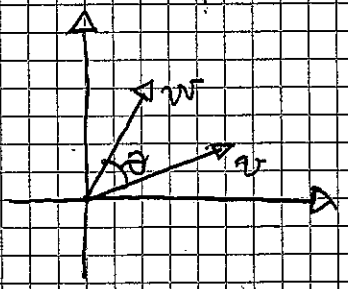
- Norma euclidea: $\|v\|_2 := \sqrt{\sum_{i=1}^{n} |v_i|^2}$
- Norma unitaria: $\|v\|_1 := \sum_{i=1}^{n} |v_i|$
- Norma infinito: $\|v\|_\infty := \max_{i=1,\dots,n} \{|v_i|\}$

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- **Spazio normato**

$d(x,y) = \|x-y\|$; $d(x+y, 0) = \|x-y\|$ → invarianza per traslazione di distanza.

- **Spazio con prodotto scalare**



$$\langle v, w \rangle = \|v\| \cdot \|w\| \cos\theta$$

DEF. Una funzione $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ si dice prodotto scalare se:

1) $\langle v, v \rangle = 0 \iff v = 0$;
2) $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle = \langle v, \lambda w \rangle \quad \forall v, w \in \mathbb{R}^n \quad \forall \lambda \in \mathbb{R}$
3) $\langle v + w, \mu \rangle = \langle v, \mu \rangle + \langle w, \mu \rangle \quad \forall v, w, \mu \in \mathbb{R}^n$
4) $\langle v, w \rangle = \langle w, v \rangle \quad \forall v, w \in \mathbb{R}^n$

OSS. $A \in \mathcal{M}(n \times n, \mathbb{R})$ simmetrica $\Rightarrow A^T = A$ ed è sempre diagonalizzabile t.c. $\lambda_i \geq 0 \ \forall i = 1, \dots, n$.

$\langle v, w \rangle_A := v^T A w$ perché $v^T A w$ è un numero!

4) $\langle v, v \rangle_A = w^T A v = (v^T A w)^T = v^T A w$

$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \quad A^T = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ → questa matrice non è simmetrica.

$(A \cdot B)^T = B^T \cdot A^T$

1) $\langle v, v \rangle_A = 0 \iff v^T A v = 0 \iff \exists U \text{ t.c. } U^{-1} A U = D = \text{diag}\{\lambda_1, \dots, \lambda_n\} \Rightarrow$
$\Rightarrow A = U^{-1} D U \iff v^T (U^{-1} D U) v = 0 \iff$
$U^{-1} = U^T \Rightarrow (v^T U^T) D (U \cdot v) = 0 \iff$
$(Uv)^T D (Uv) = 0 \iff$
$\langle Uv, Uv \rangle_D = 0 \iff$
$\sum_{i=1}^{n} \lambda_i |(Uv)_i|^2 = 0 \iff Uv = 0 \iff v = 0$ perché U è invertibile

- $\forall \langle \cdot,\cdot \rangle \; \exists A$ simmetrica e $\{\lambda_i\} > 0$ t.c. $\langle v, w \rangle = v^T A w$
- Dato $\langle \cdot,\cdot \rangle$ su $\mathbb{R}^n$ $\|v\| := \sqrt{\langle v, v \rangle}$ $\Rightarrow$ dato un prodotto scalare posso definire la metrica

Esempi

- prodotto scalare euclideo: $\langle v, w \rangle := \sum_{i=1}^{n} v_i w_i = v^T Id\, w$

$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{\sum_{i=1}^{n} |v_i|^2}$ norma euclidea

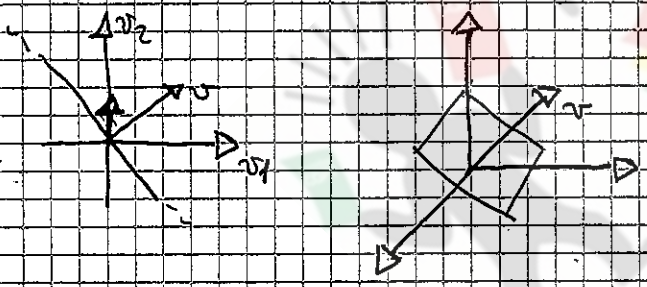
- $\|\cdot\|_1$ e $\|\cdot\|_\infty$ non derivano da un prodotto scalare.

Se $v, w \in \mathbb{R}^2$ $\langle v, w \rangle = v^T Id\, w = \|v\| \cdot \|w\| \cdot \cos\theta$

Definizione: Dato $v \in \mathbb{R}^n$, si dice che $w \in \mathbb{R}^n$ è ortogonale a v se $\langle v, w \rangle = 0$

e $v^\perp := \{w \in \mathbb{R}^n / \langle v, w \rangle = 0\} = \{w \in \mathbb{R}^n / v^T A w = 0\}$ $v \neq 0$

PROP: $v^\perp$ è un sottospazio vettoriale di $\mathbb{R}^n$ di dimensione $n-1$ se $v \neq 0$ e di dimensione n se $v = 0$.



PROP Disuguaglianza di Cauchy-Schwarz

Sia $(\mathbb{R}^n, \langle \cdot,\cdot \rangle)$ e $\|v\| = \sqrt{\langle v, v \rangle}$ $\Rightarrow$ $\forall v, w \in \mathbb{R}^n$, $|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$ e

$|\langle v, w \rangle| = \|w\| \cdot \|v\| \Leftrightarrow \exists \lambda \in \mathbb{R}$ t.c. $v = \lambda w$

Dim: Scriviamo $\langle v + \lambda w, v + \lambda w \rangle \geq 0$ $\forall v, w \in \mathbb{R}^n$ $\forall \lambda \in \mathbb{R}$

$= \langle v, v + \lambda w \rangle + \langle \lambda w, v + \lambda w \rangle = \langle v, v \rangle + \langle v, \lambda w \rangle + \langle \lambda w, v \rangle + \langle \lambda w, \lambda w \rangle$

$= \langle v, v \rangle + \lambda \langle v, w \rangle + \lambda \langle w, v \rangle + \lambda^2 \langle w, w \rangle =$

$= \langle v, v \rangle + 2\lambda \langle v, w \rangle + \lambda^2 \langle w, w \rangle \geq 0 \Leftrightarrow \Delta \leq 0$ $\Delta = 4\langle v, w \rangle^2 - \alpha$

$\forall \lambda \in \mathbb{R}$

$\Delta = |\langle v, w \rangle|^2 - \|w\|^2 \cdot \|v\|^2 \leq 0$

$\Delta < 0 \Leftrightarrow |\langle v, w \rangle| < \|v\| \cdot \|w\| \Leftrightarrow P(\lambda) > 0$

$\Delta = 0 \Leftrightarrow |\langle v, w \rangle| = \|v\| \cdot \|w\|$

$\Updownarrow$

$\exists\, \lambda_0$ t.c. $P(\lambda_0) = 0 = \langle v + \lambda_0 w, v + \lambda_0 w \rangle \Leftrightarrow v + \lambda_0 w = 0$ □

Esercizi

1) Calcolare d_1, d_2 e d_∞ tra $P = (1, 3, 0)$ e $Q = (2, 0, 1)$

$$d_1(P,Q) = \sum_{i=1}^{3} |P_i - Q_i| = |1-2| + |3-0| + |0-1| = 5$$

$$d_2(P,Q) = \sqrt{|1-2|^2 + |3-0|^2 + |0-1|^2} = \sqrt{11}$$

$$d_\infty(P,Q) = \max\{|1-2|, |3-0|, |0-1|\} = 3$$

2) Calcolare $\|\cdot\|_{1,2,\infty}$ di $(\vec{P-Q})$ dove $P = (2,1,1,0)$, $Q = (0,0,1,3)$

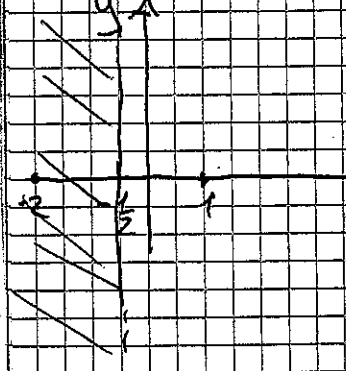
$v = (\vec{P-Q}) = (2, 1, 0, -3)$

$$\|v\|_1 = |2| + |1| + |0| + |-3| = 6$$

$$\|v\|_\infty = \max\{|2|, |1|, |0|, |-3|\} = 3$$

$$\|v\|_2 = \sqrt{|2|^2 + |1|^2 + |0|^2 + |-3|^2} = \sqrt{14}$$

3) $X = \{P = (x,y) \in \mathbb{R}^2 \mid d_2(P,(1,0)) \geq d_2(P,(-2,0))\}$



$$d_2(P,(1,0)) = \sqrt{(x-1)^2 + y^2}$$

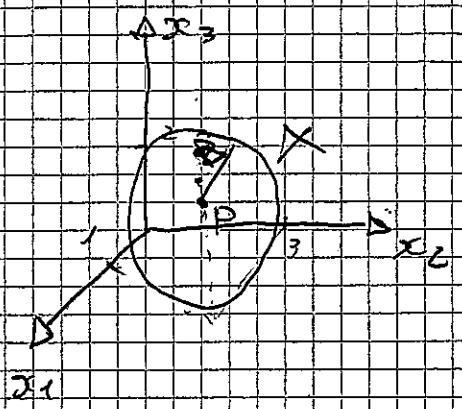
$$d_2(P,(-2,0)) = \sqrt{(x+2)^2 + y^2}$$

$$\sqrt{(x-1)^2 + y^2} \geq \sqrt{(x+2)^2 + y^2} \Rightarrow (x-1)^2 \geq (x+2)^2 \Leftrightarrow$$

$$\Leftrightarrow x \leq -\frac{1}{2}$$

4) $P=(1,3,2)$ $\quad X=\{\underline{x}\in\mathbb{R}^3 : d_2(\underline{x},P)<2\}$

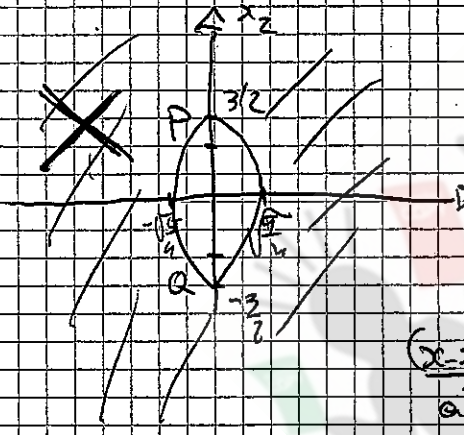
$$d_2(\underline{x},P)=\sqrt{(x_1-1)^2+(x_2-3)^2+(x_3-2)^2}<2$$



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5) $P=(0,1)$, $Q=(0,-1)$ $\quad X=\{\underline{x}\in\mathbb{R}^2 : d(\underline{x},P)+d(\underline{x},Q)\geq 3\}$

$$\sqrt{x_1^2+(x_2-1)^2}+\sqrt{x_1^2+(x_2+1)^2}\geq 3$$



per $x_1=0$ $\quad |x_2-1|+|x_2+1|=3$

$x_2=\frac{3}{2}$ $\qquad x_2=-\frac{3}{2}$

per $x_2=0$ $\quad \sqrt{x_1^2+1}+\sqrt{x_1^2+1}=3$

$$2\sqrt{x_1^2+1}=3 \Rightarrow x_1=\pm\sqrt{\frac{5}{4}}$$

$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1 \Rightarrow$ eq. dell'ellisse

$$\frac{x_1^2}{\frac{5}{4}}+\frac{x_2^2}{\frac{9}{4}}=1 \Rightarrow \frac{4}{5}x_1^2+\frac{4}{9}x_2^2=1 \Rightarrow$$

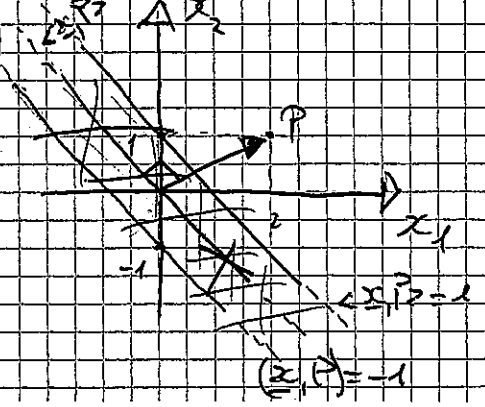
$$\Rightarrow \left[\frac{4}{5}x_1^2+\frac{4}{9}x_2^2\geq 1\right] \Rightarrow X$$

6) $P=(2,1)$ $\quad X=\{|\langle \underline{x},P\rangle|\leq 1\}$

$\langle \underline{x},P\rangle = 2\cdot x_1+1\cdot x_2 = 2x_1+x_2$

$|\langle \underline{x},P\rangle|\leq 1 \Rightarrow |2x_1+x_2|\leq 1$ $\quad 2x_1+x_2\leq 1$ $\quad 2x_1+x_2\geq -1$

$\underline{x}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$ $\quad \langle \underline{x},P\rangle=0$



$2x_1+x_2=0$ $\quad \langle \underline{x},P\rangle=0$

$2x_1+x_2=1$

$2x_1+x_2\leq 1$

$2x_1+x_2\geq -1$

$-1\leq 2x_1+x_2\leq 1$